Overview on Nonlinear Kinetic Instability

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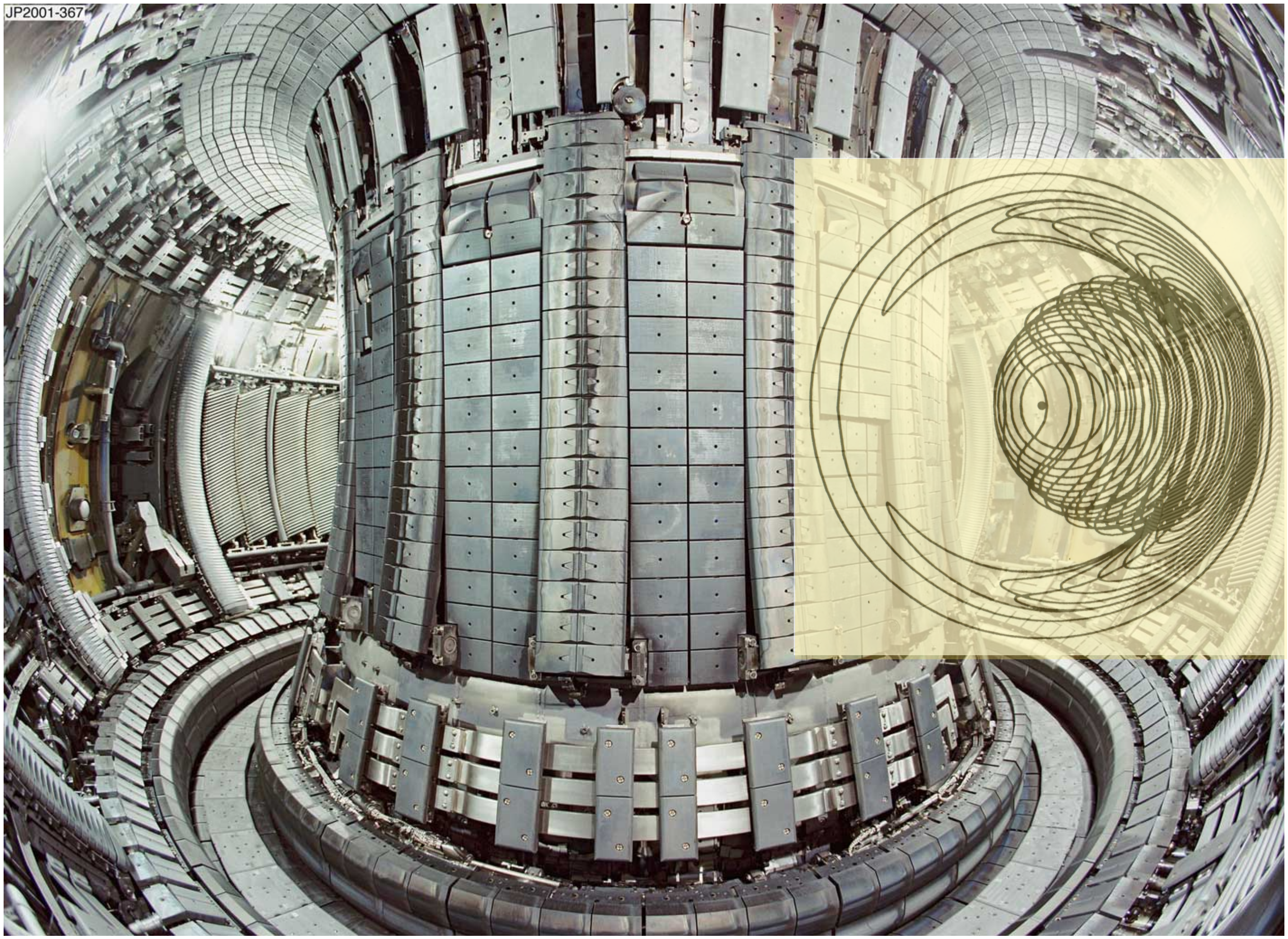
Joint European Tokamak (JET)



Fusion Motivation

Species of interest: **Alpha particles in burning plasmas**

NBI-produced fast ions

ICRH-produced fast ions

Others…

concern: **Alfvén eigenmodes (TAEs) with global spatial structure may cause global losses of fast particles**

Important insight: **Only resonant particles affected by low-amplitude modes**

Transport Mechanisms

Neoclassical: **Large excursions of mirror trapped particles (banana orbits) + collisional mixing**

$$\nu_{sc\alpha} \simeq \nu_{scth}\left(T_i / E_\alpha\right)^{3/2}, \quad \tau_{sl} \simeq \left(\frac{1}{\nu_{scth}}\right)\sqrt{\frac{M_i}{m_e}}, \quad D_\alpha \simeq D_{thcl}\left(T_i / E_\alpha\right)^{12}$$

Convective: **Transport of phase-space holes and clumps by modes with frequency chirping (Matt Lilley lecture tomorrow)**

Quasilinear : **Phase-space diffusion over a set of overlapped resonances**

Important Issue: **Individual resonances are narrow. Can they affect every particle in phase space?**

Near-threshold Nonlinear Regimes

- **Why study the nonlinear response near the threshold?**
  - Typically, macroscopic plasma parameters evolve slowly compared to the instability growth time scale
  - Perturbation technique is adequate near the instability threshold
  - Control of plasma burn is necessary to prevent excursions above threshold conditions

- **Single-mode case:**
  - Identification of the soft and hard nonlinear regimes is crucial to determining whether an unstable system will remain at marginal stability
  - Bifurcations at single-mode saturation can be analyzed
  - The formation of long-lived coherent nonlinear structure is possible (Lilley, tomorrow's lecture)

- **Multi-mode case:**
  - Multi-mode scenarios with marginal stability (and possibly transport barriers) are interesting
  - Resonance overlap can trigger hard nonlinear regime

Key Element in Theory

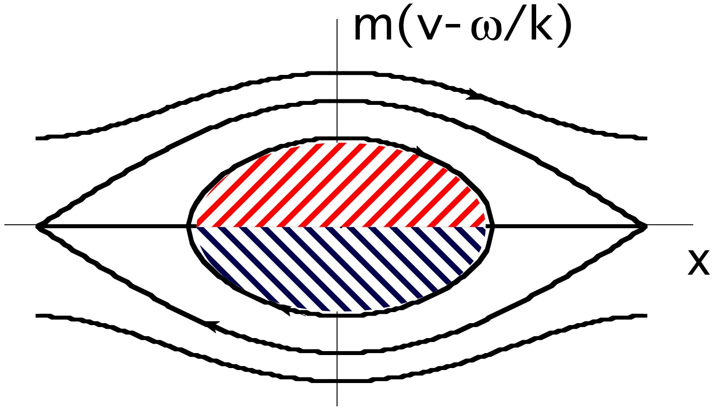
Interaction of energetic particles with unstable waves

- Pendulum equation for particles in an electrostatic wave:

$$\frac{d^2}{dt^2}\left(kx-\omega t\right)+\omega_b^2\sin(kx-\omega t)=0;\quad \omega_b^2=\frac{ekE}{m}$$

- Wave-particle resonance condition: $\omega - k\upsilon = 0$

- Phase space portrait in the wave frame:



This equation is the basis of the low amplitude nonlinear resonant response of nearly all particles confined in a tokamak and allows simple 1-D theory to provide the basic structure for describing the 3-D problem

Particle Orbits and Resonances

- Unperturbed particle motion preserves three quantities:
  - Toroidal angular momentum (P_φ)
  - Energy (E)
  - Magnetic moment (μ)
  - Poloidal Angular Momentum $P_\theta(E, P_\varphi, \mu)$ constructed

- Unperturbed motion is periodic in three angles and it is characterized by three frequencies:
  - Toroidal angle (φ) and toroidal transit frequency (ω_φ)
  - Poloidal angle (θ) and poloidal transit frequency (ω_θ)
  - Gyroangle (ψ) and gyrofrequency (ω_ψ)

- Wave-particle resonance condition:

$$\omega - n\omega_\varphi\left(\mu; P_\varphi; E\right) - l\omega_\theta\left(\mu; P_\varphi; E\right) - s\omega_\psi\left(\mu; P_\varphi; E\right) = 0$$

The quantities n, l, and s are integers with $s = 0$ for low-frequency modes.

Wave-Particle Lagrangian(ω<<ω_{ci})

- Perturbed guiding center Lagrangian for low frequency waves ($\omega << \omega_{ci}$) μ constant :

$$L = \sum_{\text{particles}} \left[P_\vartheta \dot{\vartheta} + P_\varphi \dot{\varphi} - H_0(P_\vartheta; P_\varphi; \mu) \right] + \sum_{\text{modes}} \dot{\alpha} A^2$$

$$+2\,\mathrm{Re} \sum_{\text{particles}} \sum_{\text{modes}} \sum_{\text{sidebands } l} A V_l\left(P_\vartheta; P_\varphi; \mu\right) \exp(-i\alpha - i\omega t + in\varphi + il\vartheta)$$

- Dynamical variables:
    - $P_\vartheta,\ \vartheta,\ P_\varphi,\ \varphi$ are the action-angle variables for the particle unperturbed motion
    - A is the mode amplitude
    - α is the mode phase
    - for low frequency waves ($\omega << \omega_{ci}$) μ constant

- Matrix element $V_l\left(P_\vartheta; P_\varphi; \mu\right) = \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left(il\omega_\theta \theta\right) q \frac{\tilde{\vec{E}} \cdot \vec{v}}{\omega A} \cong \left\langle q \frac{\tilde{\vec{E}} \cdot \vec{v}}{\omega A} \right\rangle_l$, where $(\tilde{\vec{E}} = \vec{E} e^{-in\phi})$

is a given function, determined by the linear mode structure

- Mode energy: $W = \omega A^2$

Resonant Reduced Hamiltonian

- For the resonant particles only one term, the resonant one, of sum needed
- We focus on this term, created from the generating function

$$F\left(\theta_{old},\phi_{old};P_{\theta nw},P_{\phi nw};t\right)=(n\phi_{old}+l\theta_{old}-\omega t)P_{\phi nw}+\theta P_{\theta nw}$$

$$\text{which yields: } \frac{\partial F}{\partial \phi_{old}}=P_{\phi old},\ \frac{\partial F}{\partial \theta_{old}}=P_{\theta old},\ \frac{\partial F}{\partial P_{\theta nw}}=\theta_{nw},\ \frac{\partial F}{\partial P_{\phi nw}}=\phi_{nw}\ ;$$

$$\mathrm{H}_{nw}\left(\theta_{nw},\phi_{nw};P_{\theta nw},P_{\phi nw},t\right)=\mathrm{H}_{old}\left(\theta_{old},\phi_{old};P_{\theta old},P_{\phi old},t\right)+\frac{\partial F}{\partial t}$$

From which we find: $P_{\theta nw}=P_{\theta old}-lP_{\phi old}/n=P_{\theta nwr},\quad P_{\phi nw}=P_{\phi old}/n,\ \ \phi_{nw}=n\phi_{old}+l\theta_{old}-\omega t$

$$H=H_0\left(P_{\theta nw}+lP_{\phi nw},nP_{\phi nw}\right)-\omega P_{\phi nw}+AV_l\left(P_{\theta nw},P_{\phi nw}\right)\exp\left(i\phi_n+i\alpha\right)+cc$$

thus θ_{nw} is ignorable coordinate, implying $P_{\theta nw}$ constant of motion

Expand Hamiltonian about resonant coordinates

$$\delta P_\phi=P_{\phi nw}-P_{\phi r},\ \delta P_\theta=P_{\theta nw}-P_{\theta r},\ \text{with } l\frac{\partial H}{\partial P_{\theta old}}+n\frac{\partial H}{\partial P_{\theta n}}=l\omega_{\theta r}+n\omega_{\phi r}=\omega$$

Producing a Hamiltonian to within a constant (does not effect dynamics)

$$\delta H=\frac{\delta P_\phi^2}{2}\frac{\partial\Omega}{\partial\delta P_\phi}\Big|_{\delta P_{\phi=0}}+2AV_l\cos\left(\phi_{nw}+\alpha\right);\qquad \omega_b^2=\left(2AV_l\frac{\partial\Omega}{\partial\delta P_\phi}\right)$$

Same pendulum Hamiltonian found for electrostatic wave

Note in terms of trapping frequency all systems have nearly the same equations

Evolution of Distribution and Wave

- Unperturbed particle motion is integrable and has canonical action-angle variables I_i and ξ_i.

- Unperturbed motion is periodic in angles ξ_1, ξ_2, and ξ_3.

- Single resonance approximation for the Hamiltonian

$$H = H_0\left(I(\Omega)\right) + 2\omega_b^2 \left(\frac{\partial P_\phi}{\partial \Omega}\right)_{\delta P_\phi = 0} \mathrm{Re}\left[\exp(i\phi + i\alpha)\right]$$

- Kinetic equation with collisions included

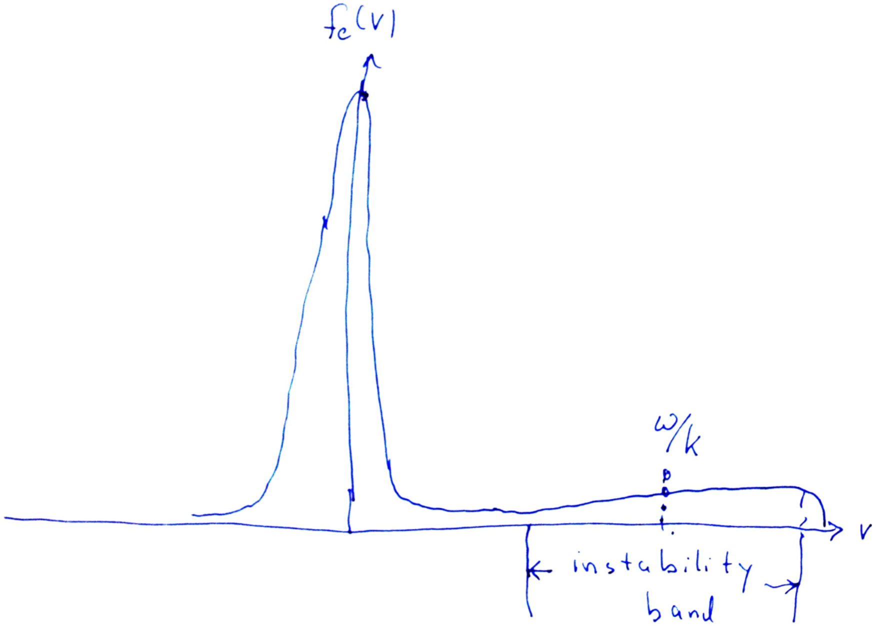
$$\frac{\partial f}{\partial t} + \Omega \frac{\partial f}{\partial \phi} - 2\,\mathrm{Re}\left[i\omega_b^2(t)\exp(i\phi + i\alpha)\right]\frac{\partial f}{\partial \Omega} = \nu_{eff}^3 \frac{\partial^2 f}{\partial \Omega^2}$$

- Equation for the mode amplitude

$$\frac{d\left(\omega_b^2 e^{i\alpha}\right)}{dt} = -\gamma_d \omega_b^2 e^{i\alpha} + \frac{i\omega}{G}\int d\Gamma V^* \exp(-i\phi) f$$

Two Physical Systems

Bump on Tail Instability



Wave extracts energy from inverted distribution

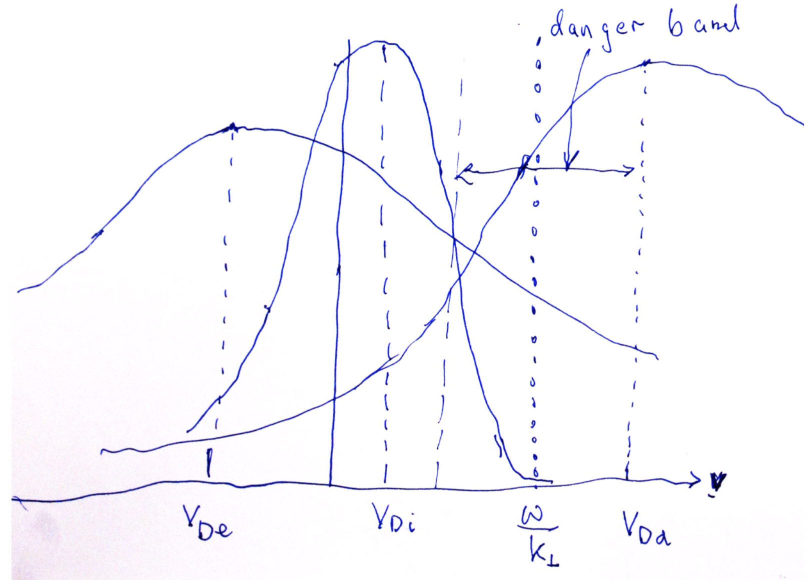
Frequency Band for "Universal" drive

equilibrium: $\underset{\sim}{j} \times \underset{\sim}{B} = \underset{\sim}{\nabla} p$

$\underset{\sim}{j} = \sum_j n_j e_j \underset{\sim}{V}_{Dj}$, $\underset{\sim}{\nabla} p = \underset{\sim}{\nabla} \sum_j p_j$

solve for V_{Dj}

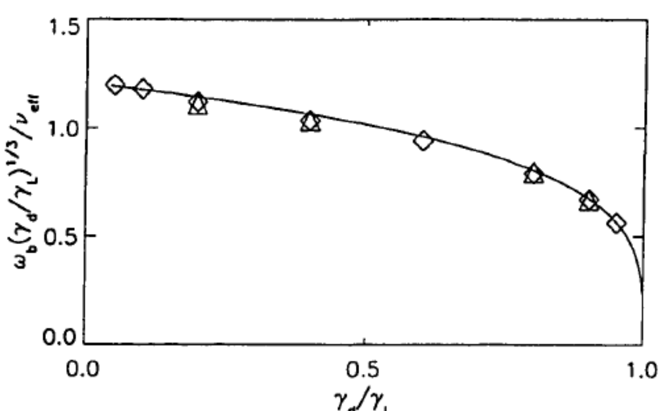
$$\underset{\sim}{V}_{Dj} \approx \frac{\underset{\sim}{b} \times \underset{\sim}{\nabla} p_j}{e_j n_j B}$$



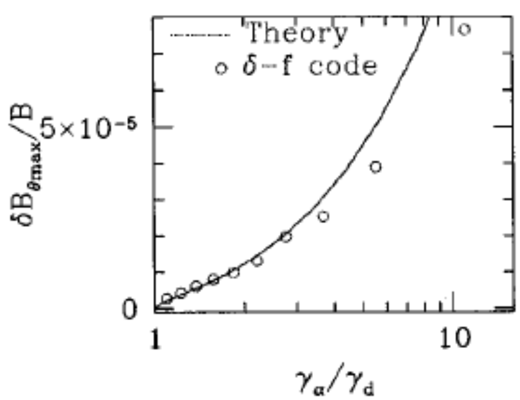
Steady state saturation Levels

Power of unstable wave lowered by nonlinear resonant response

1. $\gamma_L = 0, \quad \dfrac{\omega_b}{\gamma_L} = 3.12$
2. $\gamma_L >> \nu_{eff}\,, \quad \omega_b^3 = 1.8\nu_{eff}^3 \dfrac{\gamma_L}{\gamma_d}$
3. $\gamma_L - \gamma_d << \gamma_L\,, \quad \omega_b = 1.2\nu_{eff}\left(1 - \dfrac{\gamma_d}{\gamma_L}\right)^{1/4}$;Marginal Stability
4. Interpolation can connect 2 and 3



Bump on tail



TAE

Basic Ingredients

- Creation of an energy inverted distribution of fast particles

Examples: Bump on Tail Instability:

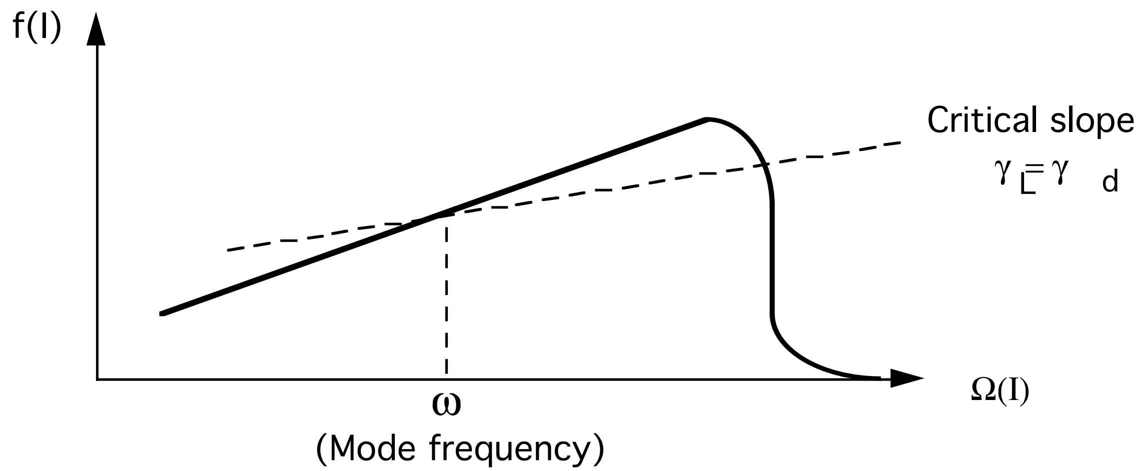
Ion Diamagnetic Drift Drive:

due to intrinsic flow velocity between different species

with waves propagating at velocities between the flows

(insert figure)

- Source to obtain steady state distribution
- Relaxation term (eg. annihilation, diffusion, drag)
- Instability drive, γ_L , due to particle-wave resonance
- Background dissipation rate, γ_d, determines the critical gradient for the instability



Steady Nonlinear State

- Wave mixes resonant particles and tends to flatten their distribution function

- Particle source feeds resonant region and maintains a finite slope, $\partial f / \partial v$, of the distribution function

- Nonlinearly reduced growth rate balances the damping rate

- IS THIS SOLUTION STABLE?

  - Yes, if $\nu_{eff} > \gamma_L - \gamma_d$

  - No, if $\nu_{eff} < \gamma_L - \gamma_d$

Mode Pulsation Scenario

1. Unstable wave grows until it flattens the distribution of resonant particles; the instability saturates when $\omega_b = \gamma_L$.

2. The excited wave damps at a rate $\gamma_d < \gamma_L$ with the distribution function remaining flat.

3. The source restores the distribution function at a rate ν_{eff}, bringing a new portion of free energy into the resonance area.

4. The whole cycle repeats.

Quasilinear Diffusion

(See Ghantos poster for more details)

$$f = \sum_k \left(f_{ko} + f_{k1} + f_{k2} \right)$$

$$\frac{\partial f_{k,j}}{\partial t} + \Omega \frac{\partial f_{k,j}}{\partial \phi} - 2\,\mathrm{Re}\left[i\omega_b^2(t)\exp(i\phi + i\alpha) \right] \frac{\partial f_{k,j-1}}{\partial \Omega} = 0$$

Iteration leads to

$$\frac{\partial \bar{f}_k}{\partial t} - \frac{\pi}{2}\frac{\partial}{\partial \Omega}\omega_b^4 \delta\left(\omega - \Omega\right)\frac{\partial \bar{f}_k}{\partial \Omega} = 0$$

Wave Equation

$$\frac{\partial \omega_b^4}{\partial t} + 2\gamma_d \omega_b^4 = \frac{\pi}{2}\beta_k \omega_b^4 \int d\Omega \delta\left(\omega - \Omega\right)\frac{\partial \bar{f}_k}{\partial \Omega}$$

Angular Momentum Conservation (with γ_d=*0*)

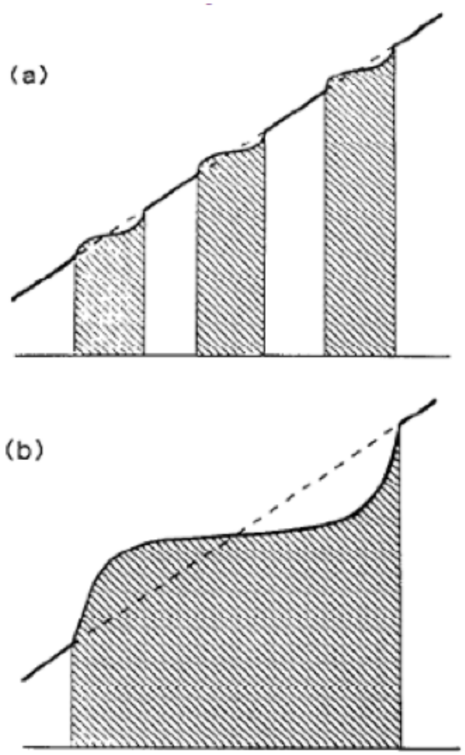
$$\beta_k \int d\Omega \Omega \bar{f}_k + \frac{\partial \omega_b^4}{\partial t} = 0$$

However resonance function needs to be broadened to account for discreet modes

$$\delta\left(\omega - \Omega\right) \to \hat{\Omega}^{-1}\left(\frac{\omega - \Omega}{\Delta\Omega}\right); \qquad \int d\Omega \hat{\Omega}^{-1}\left(\frac{\omega - \Omega}{\Delta\Omega}\right) = 1; \qquad \Delta\Omega = a\omega_b + b\nu_{eff} + \gamma$$

Momentum conservation retained; a and b chosen to best match analytic answers

Overlapped Modes Release More Energy per Mode than Discrete Modes



Estimate of Wave Energy (WE) of Discrete Mode

\# *modes* $\equiv N_{dsc} \leq \dfrac{\omega}{\gamma_L}$

WE $\approx (\gamma_L/\omega)^3 \; N_{dsc} \times$ *Strd Engtc Prtcl Enrgy* $\leq \gamma_L^2/\omega^2$

always small even when densely pact

Estimate of maximum Wave Energy (WE) of overlapped modes

(can be comparable to stored energy Though cannot be sustained)

will lead to dramatic avalanche and a slow restoration

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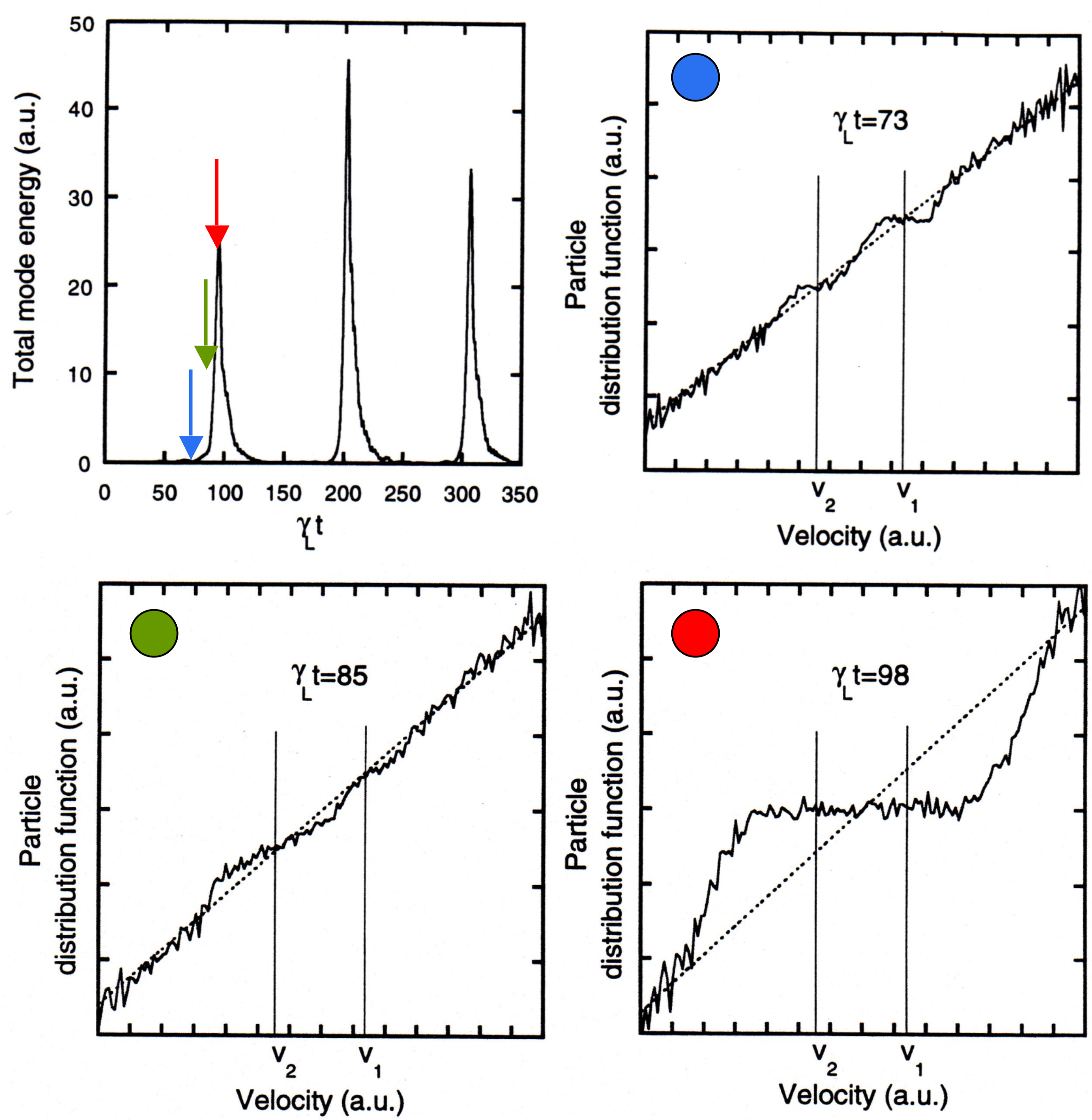
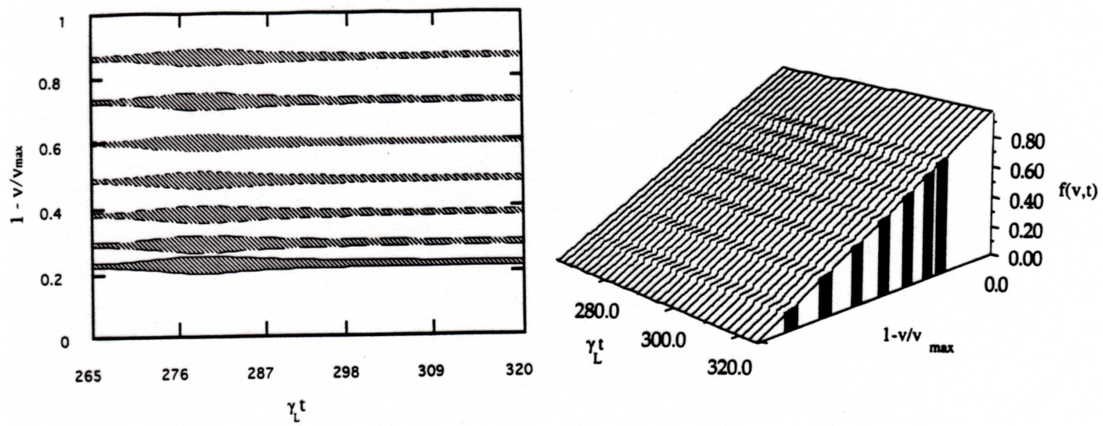
Effect of Resonance Overlap



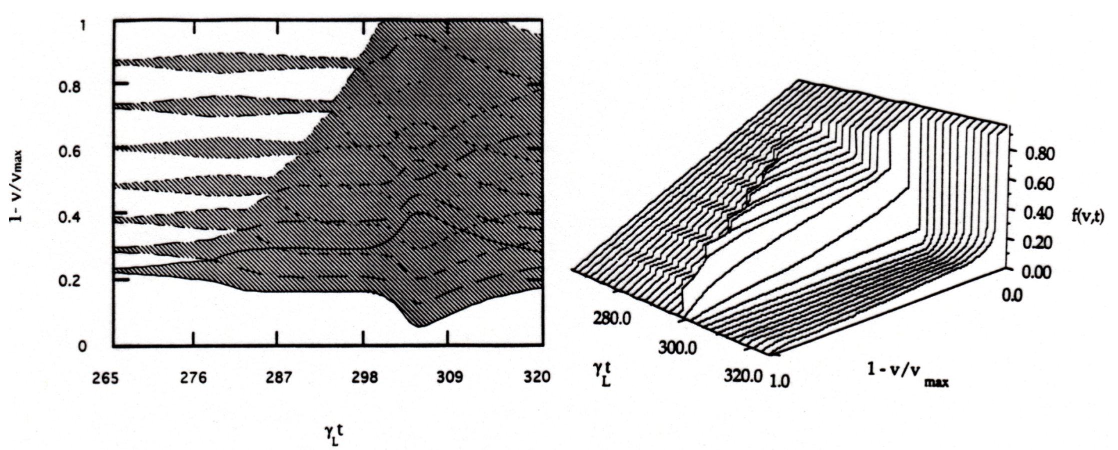
The overlapped resonances release more free energy than the isolated resonances

What Happens with Many Modes

Benign superposition of isolated saturated modes when resonances do not overlap



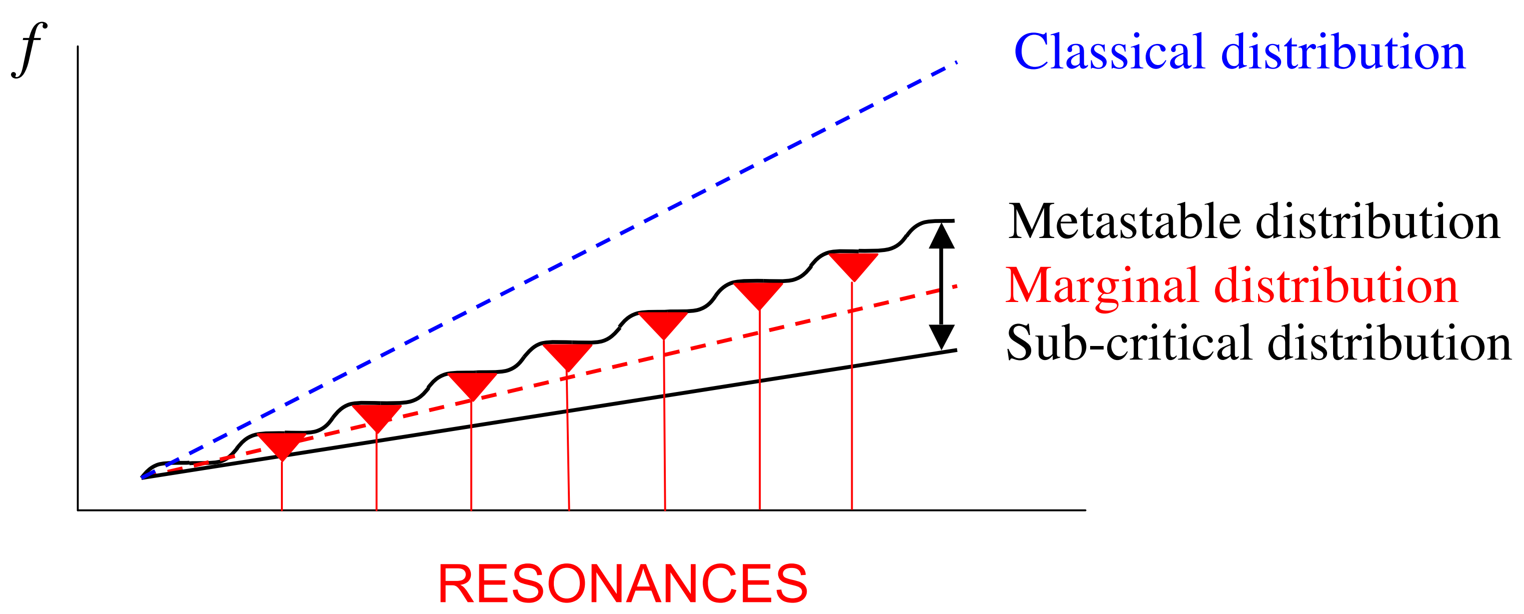
Enhanced energy release and global quasilinear diffusion when resonances overlap



Intermittent Quasilinear Diffusion

A weak source (with insufficient power to overlap the resonances) is unable to maintain steady quasilinear diffusion

Bursts occur near the marginally stable case



Summary

1. For resonant particle dynamics of low amplitude waves nonlinear theory of all physical systems in nearly identical (bump-on-tail) to TAE in tokamak
2. Discrete mode saturation levels can be calculated in various regimes and accurate interpolation methods connect results
3. Quasilinear theory has been altered to treat both separated mode case and resonance overlap case
4. Scenarios for various non-steady responses established
5. Foundations have been established for quantitative quasi-linear code treating discrete or overlapped modes
6. Interesting chirping phenomenon will be discussed in forthcoming lecture by Lilley

Finis